Stability of Hořava-Witten Spacetimes

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Abstract

Hořava-Witten spacetimes necessarily include two branes of opposite tension. If these branes are BPS we are led to a puzzle: a negative tension brane should be unstable as it can loose energy by expanding, whereas a BPS brane should be stable as it resides at a minimum of the energy. We provide a detailed analysis of the energy of such braneworld spacetimes in 5 dimensions. This allows us to show by a non-perturbative positive energy theorem that Hořava-Witten spacetimes are stable, essentially because the dynamics of the branes is entirely accounted for by the behaviour of the bulk fields. We also perform an ADM perturbative Hamiltonian analysis at quadratic order in order to illustrate the stability properties more explicitly.

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1 Introduction

Hořava-Witten (HW) theory links 11-dimensional supergravity on the orbifold $S^1/\mathbb{Z}_2$ with strongly coupled heterotic $E_8 \times E_8$ string theory [1]. It suggests that as one probes to higher energy, our 4-dimensional world first goes through an intermediate regime where the orbifold dimension becomes visible, the universe thus appearing 5-dimensional with two boundary branes [2,3]. One of these branes holds us while the second one could hold other matter that would appear "dark" to us. Only at energies of the order of the string scale would the universe look 11-dimensional.

The intermediate 5-dimensional energy regime has led to a large number of new cosmological models, coming under the general name of "braneworlds". We shall refer to such dual braneworld spacetimes generically as HW spacetimes.[1] One of the distinguishing features of such HW spacetimes is that their topology is a line element times a non-compact space, with two branes residing at the boundaries of the line element. Usually the line element is realised as the orbifold $S^1/\mathbb{Z}_2$, which can thus also be viewed in the "upstairs" picture as a circle with $\mathbb{Z}_2$ identifications. Another is that one brane has positive tension while the other has the opposite negative tension, owing to the fact that the orbifold direction is compact. By construction, these spacetimes are supersymmetric and would thus seem to be stable. However, the negative tension brane would appear to give rise to "ballooning" modes, *i.e.* one would expect that it could loose energy by expanding, thus becoming unstable.[2] In this paper we aim to resolve this issue by providing a comprehensive analysis of the energy and the fluctuations of braneworld solutions. So, the main question addressed in this paper is — which way does it go? Are supersymmetric HW spacetimes stable or unstable? In the end, we will answer this question in the affirmative.

We are going to consider perturbations about background solutions of the following type:

$$ds_5^2 \;=\; e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-8A(y)}dy^2 \;, \tag{1}$$

$$\phi \;\sim\; lnH(y), \;\; H = k|y| + c \;, \tag{2}$$

where ϕ is the scalar field supporting the branes and H is a linear harmonic function. The

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[1]Explicit models have been considered as supergravity brane solutions in $D = 5$ after dimensional reduction from $D = 11$ in Ref. [3] and in the Randall-Sundrum scenarios based on branes in AdS space in Refs [4]. A review from a cosmological viewpoint is given in Ref. [5].

[2]Indeed, at Stephen Hawking's 60th birthday conference, Brandon Carter famously exhibited this mode in a talk warning that one should pay careful attention to the possibility of instabilities in braneworld spacetimes, putting the lecturer's pointer stick under compression, and inadvertently snapping it in two. This paper is thus a reply to Brandon's concern.

$\mathbb{Z}_2$ identification appears as a symmetric kink in the harmonic function at the location of the branes. Intuitively, there are two basic types of motion that the branes can perform: a centre-of-mass motion of the set of two branes and a relative motion of the branes with respect to each other. The latter is described by the radion mode $r(x^\mu)$ which we can write as

$$ds_5^2 = e^{r(x^\rho)+2A(y)}g_{\mu\nu}^{(4)}dx^\mu dx^\nu + e^{-2r(x^\rho)-8A(y)}dy^2 \ , \tag{3}$$

$$\phi \sim lnH(y) + r(x^\rho) \ . \tag{4}$$

One can consistently truncate the theory to a system containing this mode coupled to 4-dimensional gravity [7, 8], the equations of motion then being

$$R_{\mu\nu}^{(4)} = c'\partial_\mu r\partial_\nu r \tag{5}$$

$$\Box^{(4)}r = 0 \tag{6}$$

where c' is a positive constant. The kinetic term for the radion mode hence appears with a positive sign, and this indicates that there should be no instability associated with it.

Let us now turn our attention to the centre-of-mass mode. Goldstone modes can be written down either by looking at diffeomorphisms moving the system relative to a fixed coordinate frame [6]. If we perform a coordinate transformation shifting the location of the wall by a constant parameter s, physically not much has happened in the original static solution (1,2). Brane solutions always break translational symmetries and a constant shift merely moves the brane relative to an arbitrary fixed reference coordinate frame. However, one can then promote the modulus s to a Goldstone mode by giving it a dependence on the worldvolume coordinates $s(x^\mu)$, thus allowing *relative* shifts of different points of the brane. The metric ansatz is then given by

$$ds_5^2 = e^{2A(y-s(x^\rho))}g_{\mu\nu}^{(4)}dx^\mu dx^\nu + e^{-8A(y-s(x^\rho))}dy^2. \tag{7}$$

We also set

$$H = k|y - s(x^\rho)| + c \ . \tag{8}$$

The centre-of-mass mode can be thought of in terms of the $\mathbb{Z}_2$ identification being made local as a function of the worldvolume coordinates. This mode therefore describes a sort of shear or warping of the HW end branes.[3] In many treatments of braneworlds this mode is

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[3]In an $S^1/\mathbb{Z}_2$ interpretation of the HW line element section of the spacetime, one might envision this as the torsional mode for a relative twisting of the coils in a child's "slinky" toy.

not discussed, under the assumption that the $\mathbb{Z}_2$ symmetry projects it out. However, only a global $\mathbb{Z}_2$ symmetry can project it out, and from the standpoint of the supergravity theory, there does not seem to be a clear motivation for making such an arbitrary truncation. Letting the $\mathbb{Z}_2$ become local as a function of the worldvolume coordinates preserves the upstairs/downstairs pictures of the HW line element while allowing for an intuitively natural mode of the HW system.

The mode described in the ansatz (7), based on the curved background solution (1,2), should be contrasted with a putative centre-of-mass mode in the 11-dimensional HW picture. In that case, the bulk spacetime is flat and thus, because of its translation invariance, shifting the bulk solution by a constant produces no Goldstone mode since translating the background makes no change in the local supergravity field values. Another way to see this is to note that the $\mathcal{N} = 1$, $D = 10$ supermultiplet on the end-brane worldvolume does not contain any scalar degrees of freedom. Thus, the very basic HW background solution in $D = 11$ spacetime consisting of flat space bounded by two $D = 10$ flat boundaries does not have a natural Goldstone mode and thus agrees with the string-theory picture of an orientifold, without a translational mode. By contrast, the HW spacetime in $D = 5$ obtained after reduction from $D = 11$ on a Calabi-Yau manifold to 5 dimensions has a curved bulk solution and translational scalar modes are in this case present for the bounding branes [3].

Including the translational mode (7,8) and the radion mode (3,4) allows for the two intuitively obvious motions of a set of two branes bounding a one-dimensional line element: they can either move together (translation) with local dependence on the worldvolume (warping) or oppositely (radion), causing the line element's length to locally expand or contract as a function of the worldvolume coordinates. Both the translational and the radion modes involve a stated dependence on the line element's $x^5 = y$ coordinate, but only the latter gives a Kaluza-Klein consistent "braneworld" reduction [7, 8]. The translational mode can be considered in isolation in a low-energy approximation, but, unlike the radion mode, it couples to higher 5-dimensional modes at the trilinear and higher orders. Accordingly, while we will find it instructive to consider the energy positivity properties of the translational and radion fluctuations (7,8) and (3,4) at the bilinear level (relevant to linearised fluctuation equations), understanding the full story requires use of a Witten-Nester positivity approach to the full $D = 5$ theory with boundaries.

We emphasise that both of the fluctuation modes (3,4) and (7,8) can be identified with fluctuations in the bulk geometry. This is an indication that the dynamics of the bulk by itself can give an accurate account of the physics of the system. In fact the $\mathbb{Z}_2$ symmetry of

the orbifold implies that the Israel matching conditions at the locations of the branes become sets of boundary conditions on the bulk fields.[4] We will be able to exploit this property to reduce the dynamical description of the branes to that of the nearby bulk spacetime. This is the key to showing that the energy of HW spacetimes is positive by deriving a Witten-Nester energy positivity theorem for the bulk.

In the next section we will review the HW solutions that we use here as representative braneworld solutions. Our basic arena for this discussion will be the $D = 5$ dimensional reduction of type IIB supergravity as detailed in [10]. Thus, the basic $D = 5$ brane solutions we will be working with are generalisations of RS solutions. We will present the full action of this setup, including the brane sources. We then proceed in section 3 to define the energy, and show that in static gauge it suffices to look at the stability properties of the bulk alone. This enables us to prove a Witten-Nester positive energy theorem in section 4 and thus establish the stability of the 5-dimensional HW spacetimes. In section 5 we use an ADM Hamiltonian approach to give an explicit calculation of the energy at quadratic order in fluctuations by way of illustration. We conclude with a discussion of our results.

2   A Supersymmetric RS Solution

We will briefly review the supersymmetric RS solution discussed in [11, 12]. In the RS models [4] the 5-dimensional space consists of AdS_5 space. When trying to embed the RS models in string theory, it is therefore necessary to look for theories which admit AdS_5 vacua upon compactification. Type IIB supergravity is known to have an $AdS_5 \times S^5$ vacuum solution, and hence 5-sphere reductions of this theory suggest themselves as the key to the problem. In the dimensional reduction one promotes the volume modulus of the 5-sphere to a dynamical field. This "breathing mode" gives rise to a potential in the uncompactified 5 dimensions. This gravity plus scalar system supports a domain wall solution, which can be identified with the (positive tension) brane in the RS2 model, after a $\mathbb{Z}_2$ identification of the background space at the location of the brane. In fact the theory in 5 dimensions can be truncated consistently to a scalar-gravity theory with the simple Lagrangian [10]:

$$\mathcal{L}_5 = \sqrt{-g}\,[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)], \tag{9}$$

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[4]This point has been emphasised in Ref. [9] in a reformulation of the delta-function contributions to the $D = 10 + 1$ HW action and supersymmetry transformations.

with

$$V(\phi) = 8m^2 e^{8\alpha\phi} - R_5 e^{\frac{16\alpha}{5}\phi} \ , \tag{10}$$

where m and R_5 are constants and $\alpha = \frac{\sqrt{15}}{12}$. Note that ϕ is not related to the dilaton in 10 dimensions, but instead represents the volume of the 5-sphere compactification.

The domain wall solution in this theory, which we shall take to be our basic example of a HW brane, is given by

$$ds_5^2 = (b_1 H^{2/7} + b_2 H^{5/7})^{1/2}\eta_{\mu\nu}dx^\mu dx^\nu + (b_1 H^{2/7} + b_2 H^{5/7})^{-2}dy^2 \ , \tag{11}$$

$$\phi = -\frac{\sqrt{15}}{7}ln(H), \ H = k|y| + c, \tag{12}$$

with $b_1 = \pm\frac{28m}{3k}$, $b_2 = \pm\frac{14}{15k}\sqrt{5R_5}$. Here k denotes the tension and y denotes the direction transverse to the brane. A second brane of opposite tension is placed at $y = \pi$, where a $\mathbb{Z}_2$ identification between $y = \pm\pi$ is made. Thus the topology of the full spacetime is $\mathbb{R}_4 \times S^1/\mathbb{Z}_2$. One has to choose $b_2 > 0$ and $b_1 < 0$ in order for there to exist a $k \to 0$ pure AdS limit, thus obtaining an RS scenario [8, 12]. Note that k positive (a "trough" harmonic function) corresponds to a negative tension brane, as can be verified using the Israel matching conditions. In order for the metric to be real, we need

$$H(y)^{\frac{3}{7}} > | \frac{b_1}{b_2} |, \tag{13}$$

and we therefore choose the integration constant c accordingly.

Coupling to Brane Actions

The domain wall solution (12) yields the following singular terms in the Einstein tensor and in the scalar field equation (all non-singular terms are denoted *Reg* and solve the bulk field equations) [13]:

$$G_{\mu\nu} = \frac{3k}{14}(2b_1 H^{-\frac{5}{7}} + 5b_2 H^{-\frac{2}{7}})(g_{55})^{-\frac{1}{2}}g_{\mu\nu}[\delta(y) - \delta(\pi - y)] + Reg \tag{14}$$

$$G_{yy} = 0 + Reg \tag{15}$$

$$\Box\phi = -\frac{2\sqrt{15}k}{7}(b_1 H^{-\frac{5}{7}} + b_2 H^{-\frac{2}{7}})(g_{55})^{-\frac{1}{2}}[\delta(y) - \delta(\pi - y)] + Reg \ . \tag{16}$$

The fact that four out of the five diagonal components of the Einstein tensor contain singular terms suggests that we might try to couple two 3-brane actions to the bulk theory, since the singular pieces correspond to a sum of two terms proportional to b_1 and b_2 respectively. Let

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us add source actions of the type

$$S_5^{3-brane} = -T \int_{M_4} d^4\sigma \Big[\frac{1}{2}\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu X^M \partial_\nu X^N g_{MN}(X)f(\phi(X)) - \sqrt{-\gamma}$$
$$+ \frac{1}{4!}\epsilon^{\mu\nu\rho\tau}\partial_\mu X^M \partial_\nu X^N \partial_\rho X^P \partial_\tau X^Q A_{MNPQ}(X)\Big] . \qquad (17)$$

Here T denotes the tension, σ^μ denote the worldvolume coordinates,[5] the $X^M(\sigma)$ functions represent the embedding of the brane in the ambient spacetime, $\gamma_{\mu\nu}(\sigma)$ is the worldvolume metric on the brane and we have allowed for an as yet unspecified coupling to the scalar field via $f(\phi(X))$. Also, $A_{(4)}(X)$ is a 4-form field that is required for consistency in a $\mathbb{Z}_2$ symmetric background: it represents the charge of the brane and is needed in order for the equation of motion resulting from varying X to be satisfied.

We also need to have kinetic terms for the $A_{(4)}$ fields. At this point, it is useful to remember that dimensional reduction of the action for the $F_{[5]}$ 5-form field strength in $D = 10$ type IIB supergravity gives rise to just such a kinetic term in $D = 5$ [10]. Taken by itself, this field strength describes no $D = 5$ continuous degrees of freedom, as a cursory review of its gauge structure reveals. It is a "theory-of-almost-nothing" field. The caveat impled by "almost" is that $F_{[5]}$ couples to the 5-sphere volume modulus ϕ, and the integration constant arising from the $F_{[5]}$ field equation gives rise to a cosmological potential for ϕ.

Actually, in the S^5 dimensional reduction of type IIB theory, there are two independent types of potential terms that arise for ϕ – one with a coefficient depending on the $F_{[5]}$ form field expectation value m, and the other with a coefficient determined by the value of the Ricci scalar on the 5-sphere R_5. There are correspondingly two distinct types of singularity structure that occur in the 3-brane solutions to this $D = 5$ theory, as we have seen in (14-16). One of these arises from the S^5 dimensional reduction of the classic D3 brane of $D = 10$ type IIB supergravity, while the other can be viewed as arising from a $\mathbb{Z}_2$ identification of two regions of $D = 10$ flat space [13]. It is convenient to introduce for this purpose a second 5-form "theory-of-almost-nothing" field strength in $D = 5$ in order to input the second potential coefficient from a separate brane source coupling in a fashion similar to the way the D3 brane charge is put in. In fact, the left and right locations of these two brane sources can be separated. Taking $i = 1, 2$ to denote the (left, right) locations, we take the first type of sources to be located at X_i^M and the second type to be at $\tilde{X}_i^M$. Both of these

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[5]With a slight abuse of notation we choose $\mu, \nu, ...$ indices to denote worldvolume directions $0, 1, 2, 3$ in anticipation of the fact that we will choose the static gauge later on where the coordinates of the brane are aligned with the coordinates of the bulk.

source actions are consistent with the general scheme for handling supersymmetric solutions in singular spaces of Ref. [14]. We take the dimensionally reduced $F_{[5]}$ of type IIB theory to be $F_{[5]} = dA_{[4]}$ and the second 5-form field strength to be $\tilde{F}_{[5]} = d\tilde{A}_{[4]}$.

Then using the Israel junction conditions to determine the coupling to the brane actions (see Appendix B), we find the brane + bulk action

$$
\begin{aligned}
S_5 &= \int d^5 x \sqrt{-g} \, [R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2 \cdot 5!} e^{-8\alpha\phi} F_{[5]}^2 + \frac{1}{4 \cdot 4!} e^{-\frac{16}{5}\alpha\phi} \tilde{F}_{[5]}^2] \\
&\quad -4m \sum_{i=1}^{2} s_i \int d^5 x \int d^4 \sigma \delta^5(x - X_{(i)}) [\sqrt{-\gamma} \gamma^{\mu\nu} \partial_\mu X_i^M \partial_\nu X_i^N g_{MN} e^{2\alpha\phi} - 2\sqrt{-\gamma} \\
&\quad + \frac{2}{4!} \epsilon^{\mu\nu\rho\sigma} \partial_\mu X^M \partial_\nu X^N \partial_\rho X^P \partial_\sigma X^Q A_{MNPQ}] \\
&\quad + \sqrt{5R_5} \sum_{i=1}^{2} s_i \int d^5 x \int d^4 \sigma \delta^5(x - \tilde{X}) [\sqrt{-\gamma} \gamma^{\mu\nu} \partial_\mu \tilde{X}^M \partial_\nu \tilde{X}^N g_{MN} e^{\frac{4}{5}\alpha\phi} - 2\sqrt{-\gamma} \\
&\quad + \frac{2}{4!} \epsilon^{\mu\nu\rho\sigma} \partial_\mu \tilde{X}^M \partial_\nu \tilde{X}^N \partial_\rho \tilde{X}^P \partial_\sigma \tilde{X}^Q \tilde{A}_{MNPQ}],
\end{aligned}
\tag{18}
$$

where $s_1 = 1$, $s_2 = -1$ give the opposing charges of the (left, right) branes of each type. In the following, we shall take the two brane types on each side of the interval to be coincident, i.e. $X_i^M = \tilde{X}_i^M$, as we are not interested here in discussing separately the dynamics of each type. However, the general action (18) will be important for us as it will allow us to discuss carefully the nature of the brane-bulk interaction and energy conservation. For completeness we give the equations of motion resulting from this action in Appendix A.

We may use the brane worldvolume reparameterization freedoms and $D = 5$ general coordinate invariance to choose a static gauge[6] where $X_i^\mu = \sigma^\mu$, $X_1^5 = 0$ and $X_2^5 = \pi$. Note that we are not fixing the physical positions of the left and right branes, just the choice of coordinates by which we designate the two branes. Their physical location within the $D = 5$ spacetime can fluctuate as a function of the bulk supergravity fields.

In the static gauge, the equations of motion for the 5-form field strengths reduce to

$$
\nabla_y (e^{-8\alpha\phi} F_{[5]}^{y\mu\nu\rho\sigma}) = 8m[\delta(y) - \delta(y - \pi)] \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma},
\tag{19}
$$

$$
\nabla_y (-\frac{5}{2} e^{-\frac{16}{5}\alpha\phi} \tilde{F}_{[5]}^{y\mu\nu\rho\sigma}) = 2\sqrt{5R_5}[\delta(y) - \delta(y - \pi)] \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} .
\tag{20}
$$

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[6]As discussed in [15], the static gauge for a two-brane system does not overfix the coordinate and reparameterization gauge freedoms.

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They have the solutions

$$F_{MNPQT} = 4me^{8\alpha\phi}\theta(y)\sqrt{-g}\,\epsilon_{MNPQT} \tag{21}$$

$$\tilde{F}_{MNPQT} = -\frac{2}{5}\sqrt{5R_5}e^{\frac{16}{5}\alpha\phi}\theta(y)\sqrt{-g}\,\epsilon_{MNPQT}\,, \tag{22}$$

where

$$\theta(y) = \begin{cases} +1 & \text{for } 0 \le y < \pi \\ -1 & \text{for } -\pi \le y < 0 \end{cases} \tag{23}$$

and we impose the upstairs-picture identification $y \sim y + 2\pi$. Note that this sign flipping for the 5-forms across the location of the brane as given in (21,22) is in agreement with what we expect for supersymmetry in a $\mathbb{Z}_2$ symmetric singular spacetime [14]. We will make the correspondence to the BKVP formalism explicit in Appendix C. We shall mainly work in static gauge in the following.

3   Energy

As explained in the introduction, the bulk fields effectively contain all the necessary knowledge about the boundary branes due to the Israel junction conditions, which are boundary conditions as a result of the $\mathbb{Z}_2$ symmetry (see Appendix B). This means that if we can show positivity of the energy in the bulk, we will have fully shown the stability of this class of spacetimes. Let us therefore proceed by first defining the energy in the bulk.

It is well known that energy is not defined in a very obvious way in any theory containing gravity [16–18]. In particular, diffeomorphism invariance implies that the theory is invariant under arbitrary time reparameterizations. Energy is defined for spacetimes that admit asymptotically a timelike Killing vector field. If we want an expression for the energy in terms of fluctuations about a background possessing a global timelike Killing vector, we need to expand the Einstein equations in terms of fluctuations about the background, and then separate out the terms of quadratic and higher orders in the fluctuations from terms that are linear in the fluctuations. The background is taken to satisfy the field equations exactly. Let us write

$$g_{MN} = g_{MN}^{(0)} + h_{MN} \tag{24}$$

$$\phi = \phi^{(0)} + \phi^{(1)} \tag{25}$$

and similarly for all other fields. Here $h_{MN} = g_{MN}^{(1)}$ is the fluctuation of the metric and the superscript denotes the order in fluctuations. So we rewrite the Einstein equations as

$$\tau_{MN} \equiv T_{MN}^{(2+\text{higher})} - G_{MN}^{(2+\text{higher})} \tag{26}$$

$$= R_{MN}^{(1)} - \frac{1}{2}g_{MN}^{(0)}g^{(0)RS}R_{RS}^{(1)} + \frac{1}{2}g_{MN}^{(0)}h^{RS}R_{RS}^{(0)} - \frac{1}{2}h_{MN}g^{(0)RS}R_{RS}^{(0)} - T_{MN}^{(1)} \tag{27}$$

where τ_{MN} is the energy-momentum pseudotensor containing the contributions due to gravitational energy.

Now τ_{MN} satisfies

$$\nabla_N^{(0)}\tau^{MN} \propto \phi^{(0),M} \times [\text{linearised } \phi \text{ field equation}] \tag{28}$$

which is non-zero in general since we cannot impose the linearised matter field equations (this would be in conflict with the fact that we are imposing the full Einstein and matter field equations). However, since our background satisfies

$$\phi^{(0),M}\xi_M^{(0)} = 0, \tag{29}$$

where $\xi_M^{(0)}$ is a timelike background Killing vector, we have it that[7]

$$(\nabla_N^{(0)}\tau^{MN})\xi_M^{(0)} = 0. \tag{30}$$

Using the defining property of the Killing vector

$$\nabla_M^{(0)}\xi_N^{(0)} + \nabla_N^{(0)}\xi_M^{(0)} = 0, \tag{31}$$

we can construct the ordinarily conserved vector density [17]

$$\nabla_N^{(0)}(\sqrt{-g^{(0)}}\tau^{MN}\xi_M^{(0)}) = \partial_N(\sqrt{-g^{(0)}}\tau^{MN}\xi_M^{(0)}) = 0 . \tag{32}$$

This then enables us to define the energy as

$$Q = \int_V dV \sqrt{-g^{(0)}}\tau^{0M}\xi_M^{(0)} , \tag{33}$$

where dV is a 4-spatial volume element. We note that for the solution (12) we have $\xi^{(0)M} = (1, \underline{0})$. We can then look at the conservation of the energy by calculating (where $i, j, k, ...$

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[7]A similar line of reasoning was advocated by Deser and Soldate in their discussion of the energy of the Kaluza-Klein monopole [18].

indices denote spatial directions)

$$\frac{\partial}{\partial t}Q = -\int dV \partial_i [\sqrt{-g^{(0)}}(G^{(1)Mi} - T^{(1)Mi})\xi_M^{(0)}] \tag{34}$$

$$= -[\sqrt{-g^{(0)}}(G^{(1)05} - T^{(1)05})\xi_0^{(0)}]_{y=0}^{y=\pi} \tag{35}$$

$$= 0 \tag{36}$$

The last line follows because $G^{(1)05}$ and $T^{(1)05}$ are continuous and odd under the $\mathbb{Z}_2$ symmetry and thus vanish at the location of the branes. The fact that they are continuous can be explained by observing that the brane energy-momentum tensor is given by

$$T_{brane}^{05} \propto \gamma^{\mu\nu}\partial_\mu X^0 \partial_\nu X^5, \tag{37}$$

as can be read off from (87) in Appendix A. Thus in the static gauge $X^\mu = \sigma^\mu$, $X^5 = 0, \pi$ we have

$$T_{brane}^{05} = 0. \tag{38}$$

This holds at every order in perturbation theory, and it shows that there are no singular contributions to the 05 Einstein equation. Moreover this shows that the bulk energy is conserved without any contribution from the brane variables.

Before proceeding, let us first give here the ADM surface form of the energy [19]. Indeed, $\tau^{0M}\xi_M^{(0)}$ can be rewritten as a total derivative, thus yielding the surface form (where we have dropped the $^{(0)}$ superscript on background fields in order to avoid cluttering the expression):

$$Q = \frac{1}{2}\int_{\partial V} d\Sigma_i(\xi_N h^{iN;0} - \xi_N h^{0N;i} + \xi^0 h^{,i} - \xi^i h^{,0} + h^{0N}\xi_{N;}{}^i - h^{iN}\xi_{N;}{}^0$$

$$+ \xi^i h^{0N}{}_{;N} - \xi^0 h^{iN}{}_{;N} + h\xi^{i;0} + \xi^0 \phi^{,i}\phi^{(1)} - \xi^i \phi^{,0}\phi^{(1)}) \tag{39}$$

where the semicolons denote covariant differentiation with respect to the background metric. In a similar way, one can define three total momentum charges associated with the spacelike Killing vectors corresponding to the spatial worldvolume translational symmetries of the background.

4 Positive Energy

As we saw in the last section, we can look at the bulk alone in order to prove the stability of the 5-dimensional HW spacetimes. Thus, if the energy can be shown to be positive at a given time, it will remain so due to the bulk field equations alone, with no contribution

from the boundary X^μ variables. In the bulk, we can rewrite our theory (9) in terms of a potential derived from a superpotential

$$\mathcal{L} = \sqrt{-g}\,[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)] \tag{40}$$

$$W(y,\phi) = \sqrt{2}(2me^{4\alpha\phi} - 5\sqrt{\frac{R_5}{20}}e^{\frac{8}{5}\alpha\phi})\theta(y) \tag{41}$$

where

$$V(\phi) = W_{,\phi}^2 - \frac{2}{3}W^2 \tag{42}$$

$$= 8m^2 e^{8\alpha\phi} - R_5 e^{\frac{16}{5}\alpha\phi} . \tag{43}$$

When extended to include fermions, the theory is invariant under the supersymmetry transformations [20]

$$\delta\psi_M \equiv \mathcal{D}_M\epsilon = [\nabla_M - \frac{1}{6\sqrt{2}}\Gamma_M W(y,\phi)]\epsilon + \text{higher order in fermions} \tag{44}$$

$$\delta\lambda = (\frac{1}{2}\Gamma^M\nabla_M\phi + \frac{1}{\sqrt{2}}W_{,\phi})\epsilon + \text{higher order in fermions} . \tag{45}$$

We can then prove positivity of the energy of a purely bosonic solution to the theory (40) at all orders using a Witten-Nester type argument as developed in several stages in Refs [21–27]. Let us define the Witten-Nester energy (we will show later on that this definition is equivalent to the one given in the previous section) by

$$E_{\text{WN}} = \int_{\partial V} *E \tag{46}$$

where the integral is taken over the boundary of the spatial volume element V, and where $*E$ is the Hodge dual of the Nester 2-form $E = \frac{1}{2}E_{MN}dx^M dx^N$, defined by

$$E^{MN} = \bar{\eta}\,\Gamma^{MNP}\mathcal{D}_P\eta - \overline{\mathcal{D}_P\eta}\,\Gamma^{MNP}\eta \tag{47}$$

where η denotes here a commuting spinor function that asymptotically tends to a background Killing spinor, $i.e.$ it satisfies

$$\mathcal{D}_M^{(0)}\eta = 0 \tag{48}$$

$$\frac{1}{2}\Gamma^M\nabla_M^{(0)}\phi^{(0)}\eta + \frac{1}{\sqrt{2}}W_{,\phi}^{(0)}\eta = 0 \tag{49}$$

asymptotically as $|x^{1,2,3}| \to \infty$. The anticommuting supersymmetry parameter appearing in the fermion transformations (44,45) is given by η times an anticommuting constant.

We can next use Gauss' Law to rewrite the energy as an integral over V (where $d\Sigma_0 = dV$)

$$E_{\text{WN}} = \int_V d\Sigma_M \sqrt{-g}\, \nabla_N E^{MN} \tag{50}$$

$$= \int_V d\Sigma_M \sqrt{-g}\, [\overline{\mathcal{D}_N \eta}\, \Gamma^{MNP} \mathcal{D}_P \eta + \bar{\eta}\Gamma^{MNP}\mathcal{D}_N \mathcal{D}_P \eta + \text{h.c.}] \tag{51}$$

The second term can be rewritten in terms of the dilatino supersymmetry transformation (45). If we now impose the Witten condition

$$\Gamma^k \mathcal{D}_k \eta = 0 \ , \tag{52}$$

and choose to foliate our spacetime in terms of spatial slices at constant times, we can express the Witten-Nester energy in terms of $\tilde{\delta}\psi_i$, $\tilde{\delta}\lambda$, which are related to $\delta\psi_i$ and $\delta\lambda$ in (44,45) by replacing the anticommuting spinor parameter ϵ by the commuting η, which is subject to the Witten condition (52):

$$E_{\text{WN}} = 2\int_V dV \sqrt{-g}\, [(\tilde{\delta}\psi_i)^\dagger \tilde{\delta}\psi_i + (\tilde{\delta}\lambda)^\dagger \tilde{\delta}\lambda] \geq 0 \ . \tag{53}$$

What remains to be done is to show that the energy definition given in (46) actually agrees with the previous definition (33). Let us expand the expression (46) in fluctuations about the background by perturbing the vielbeins $e^P{}_a = e^{(0)P}{}_a + \frac{1}{2}h^P{}_a$, where $a, b, ...$ denote tangent space indices. For the spin connection we find

$$\omega^{(1)}_{Pab} = \frac{1}{2}(h_{Pa;b} - h_{Pb;a}) \tag{54}$$

and we also expand

$$\mathcal{D}_P \eta = \frac{1}{4}\omega^{(1)}_{Pab}\Gamma^{ab}\eta - \frac{1}{6\sqrt{2}}\Gamma_P W^{(0)}_{,\phi}\phi^{(1)}\eta - \frac{1}{12\sqrt{2}}h_{Pa}\Gamma^a W^{(0)}\eta + ... \tag{55}$$

for a Killing spinor η. The energy (46) can then be written as

$$E_{\text{WN}} = \int_{\partial V} \quad [\frac{1}{4}\bar{\eta}\Gamma^M \eta (h^{;N} - h_P{}^{N;P}) - \frac{1}{4}\bar{\eta}\Gamma^N \eta (h^{;M} - h_P{}^{M;P})$$

$$+ \frac{1}{4}\bar{\eta}\Gamma^P \eta (h_P{}^{N;M} - h_P{}^{M;N})$$

$$- \frac{1}{12\sqrt{2}}\bar{\eta}(\Gamma^{MN}h + \Gamma^{NP}h_M{}^P + \Gamma^{PM}h_P{}^N)W(\phi)\eta$$

$$- \frac{1}{2\sqrt{2}}\bar{\eta}\Gamma^{MN}W_{,\phi}\phi^{(1)}\eta]d\Sigma_{MN} + \text{h.c.} \tag{56}$$

We now relate Killing spinors to Killing vectors by

$$\xi^{(0)M} = \bar{\eta}\Gamma^M \eta. \tag{57}$$

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Using relations (48,49) one can see that the Witten-Nester surface expression for the energy is exactly equivalent to the ADM type formula (39), with the terms of the form $\bar{\eta}\Gamma^{MN}W\eta$ proportional to terms involving $\xi^{(0)N;M}$, and the terms of the form $\bar{\eta}\Gamma^{MN}W_{,\phi}\eta$ responsible for the $\xi^{(0)[M}\phi^{(0),N]}$ contributions. Thus we have verified that our two definitions of the energy are in agreement. We can therefore conclude that the energy is manifestly positive and that the HW spacetimes are stable despite the presence of brane sources of negative tension[8].

5 Positivity at Quadratic Order

The Witten-Nester argument just presented is very powerful, albeit rather non-intuitive. We feel that it is sometimes good to also have a more concrete, although less general, argument and therefore we wish to show here, by means of explicit illustration, that the energy is positive at quadratic order in perturbation theory [29]. In fact, any potential instability would presumably manifest itself already at this order, and therefore stability at quadratic order can already be seen as a strong indication of stability to all orders. For our calculations in this section we will use the ADM Hamiltonian approach [30], as it is best suited for this purpose. We perform an explicit $(1+4)$ decomposition of the metric, choosing spacetime to be foliated along constant time slices. The metric can be written as

$$ds^2 = (N_i N^i - N^2)dt^2 + 2N_i dx^i dt + g_{ij}dx^i dx^j \ , \tag{58}$$

where N is the lapse function, N^i the shift function and $g_{ij} = {}^{(5D)}g_{ij}$ (see [31] for details). Indices are lowered and raised by g_{ij} and its inverse g^{ij} (which does not equal ${}^{(5D)}g^{ij}$ in general!). A dot on top of a quantity denotes a time derivative, while $|$ denotes covariant differentiation with respect to the 4-dimensional metric. The embedding of the 4-dimensional hypersurface in the 5-dimensional bulk spacetime is characterised by the extrinsic curvature K_{ij}, given as

$$K_{ij} = \frac{1}{2N}(-\dot{g}_{ij} + N_{i|j} + N_{j|i}) \ . \tag{59}$$

The "momentum" conjugate to the metric is defined as

$$\pi^{ij} \equiv \frac{\delta \mathcal{L}}{\delta \dot{g}_{ij}} = -g^{\frac{1}{2}}(K^{ij} - g^{ij}K) \ , \tag{60}$$

and the momentum P conjugate to the scalar field ϕ reads

$$P \equiv \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \frac{g^{\frac{1}{2}}}{N}(\dot{\phi} - N^i \phi_{|i}) \ . \tag{61}$$

---

[8]We note that the analogous four dimensional Schwarzschild solution with negative mass parameter has recently been shown to be stable subject to linearised perturbations of finite total energy in Ref. [28]

In terms of these new variables, the action can be rewritten [32]

$$
\begin{aligned}
S \;=\; & \int dt d^4x \{ \pi^{ij} \dot{g}_{ij} + P \dot{\phi} \\
& - N g^{-\frac{1}{2}} [\pi^{ij} \pi_{ij} - \frac{1}{3}\pi^2 - g(^{(4D)}R - V(\phi)) + \frac{1}{2}P^2 + \frac{1}{2}g g^{ij}\phi_{|i}\phi_{|j}] \quad (62) \\
& - N_i[-2\pi^{ij}{}_{|j} + \phi^{|i}P] \}
\end{aligned}
$$

We see that N and N_i act as Lagrange multipliers, imposing respectively the constraints

$$
\pi^{ij}\pi_{ij} - \frac{1}{3}\pi^2 - g(^{(4D)}R - V(\phi)) + \frac{1}{2}P^2 + \frac{1}{2}g g^{ij}\phi_{|i}\phi_{|j} = 0 \tag{63}
$$
$$
-2\pi^{ij}{}_{|j} + \phi^{|i}P = 0 . \tag{64}
$$

At background order, these constraints are of course satisfied by the solution (12), with

$$
{}^{(4D)}R = \frac{1}{2}g^{(0)ij}\phi^{(0)}_{|i}\phi^{(0)}_{|j} + V(\phi) \tag{65}
$$

and

$$
0 = P^{(0)} = \pi^{(0)ij} = N^{(0)i} = \dot{\phi}^{(0)} . \tag{66}
$$

We impose these constraints at linear order, where they read

$$
-{}^{(4D)}R^{(1)} + \frac{\partial V}{\partial \phi}\phi^{(1)} - \frac{1}{2}h^{ij}\phi^{(0)}_{|i}\phi^{(0)}_{|j} + g^{(0)ij}\phi^{(0)}_{|i}\phi^{(1)}_{|j} = 0 \tag{67}
$$
$$
2\pi^{(1)ij}{}_{|j} = \phi^{(0)|i}P^{(1)} , \tag{68}
$$

with

$$
{}^{(4D)}R^{(1)} = h^{ij}{}_{|ji} - h^i{}_{i|j}{}^j - h^{ij(4D)}R^{(0)}_{ij}. \tag{69}
$$

If we write the action in the form

$$
S = \int dt d^4x \, [\pi^{ij}\dot{g}_{ij} + P\dot{\phi} - NH - N_i H^i], \tag{70}
$$

we can read off the Hamiltonian

$$
\mathcal{H} = \int d^4x \, [NH + N_i H^i] . \tag{71}
$$

To second order, subject to the constraints at linear order, we then find

$$
\begin{aligned}
\mathcal{H}^{(2)} \;=\; & \int d^4x \{ N^{(0)}g^{(0)-\frac{1}{2}}[\pi^{(1)ij}\pi^{(1)}_{ij} - \frac{1}{3}\pi^{(1)2} + \frac{1}{2}P^{(1)2}] \\
& + N^{(0)}g^{(0)\frac{1}{2}}[\frac{1}{4}h^{ij|k}h_{ij|k} + \frac{1}{4}h^i{}_{i|}{}^j h^k{}_{k|j} - \frac{1}{2}h^{ij}{}_{|j}h_{ik|}{}^k - \frac{1}{12}h^{ij}h_{ij}(V + \frac{1}{2}g^{(0)kl}\phi^{(0)}_{,k}\phi^{(0)}_{,l})] \\
& + N^{(0)}g^{(0)\frac{1}{2}}[\frac{1}{2}\frac{\partial^2 V}{\partial \phi^2}\phi^{(1)2} + \frac{1}{2}(\phi^{(1)|i} - h^{ij}\phi^{(0)}_{|j})(\phi^{(1)}_{|i} - h_i{}^k\phi^{(0)}_{|k})] \} . \tag{72}
\end{aligned}
$$

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It is straightforward to verify that the second order Hamiltonian is conserved in time subject to the linearised field equations. We can see that its expression exhibits terms of various signs. However, before drawing any premature conclusions, we must remember that no gauge choice has yet been imposed, and therefore there is still a large amount of ambiguity in the precise meaning of the above expression. We have 5 gauge choices at our disposal. Let us choose:

$$h^{ij}{}_{|j} = 0, \tag{73}$$

$$P^{(1)2} = \frac{2}{3}\pi^{(1)2} + g^{(0)} \mid \frac{\partial^2 V}{\partial \phi^2} \mid \phi^{(1)2} . \tag{74}$$

These gauge choices are non-conflicting and are independent of each other. Now note that the requirement $V + \frac{1}{2}g^{(0)ij}\phi^{(0)}_{|i}\phi^{(0)}_{|j} \leq 0$ for the background translates into

$$\frac{3k^2}{196H}(16b_1^2 H^{-\frac{3}{7}} + 20b_1 b_2 - 5b_2^2 H^{\frac{3}{7}}) \leq 0 , \tag{75}$$

and can be verified to be always satisfied if the reality condition (13) on the metric is imposed. Therefore, our expression for the second order Hamiltonian is manifestly positive, indicating that this system is stable, despite the presence of the negative-tension domain wall.

6   Discussion

We have considered the energy and the stability of Hořava-Witten spacetimes, and we have shown that it is essentially the unbroken supersymmetry of the theory and of the static background solution that guarantees their stability: the supersymmetric background acts like a "vacuum" whose energy bounds from below that of neighbouring perturbations. This happens despite the presence of negative tension branes which might have been thought to give rise to unstable modes. In fact, the dynamics of the branes can be studied *via* the dynamics of the bulk fields at and near the location of the boundaries. This is possible because the world volume *local* $\mathbb{Z}_2$ symmetry of the solutions that we consider gives rise to boundary conditions relating brane and bulk.

In fact, one might wonder what would happen if one were to relax altogether the requirement of a local $\mathbb{Z}_2$ symmetry. Perturbations of this type would change the topology of the solutions considered to $S^1 \times \mathbb{R}^4$, and such perturbations are not included in the analysis of this paper. In that case, the matching conditions at the branes would not simply reduce to boundary conditions and our expression for the bulk energy would not be a separately

conserved quantity anymore. The threat of unstable modes from the negative tension brane is thus likely to become much more substantial in that case.

Acknowledgments

Appendix A

We denote spacetime coordinates by $(x^0, x^1, x^2, x^3, y) = (0, 1, 2, 3, 5)$. Indices $M, N, P, ...$ run over all coordinate values; $a, b, ...$ denote flat tangent space indices, whereas $\mu, \nu, ...$ run over $0, 1, 2, 3$ and $i, j, ...$ are purely spatial and take the values $1, 2, 3, 5$. Our metric has signature $(-, +, +, +, +)$ and our conventions for gravity are

$$R_{MN} = \partial_P \Gamma^P_{MN} - \partial_M \Gamma^P_{NP} + \Gamma^P_{MN}\Gamma^L_{PL} - \Gamma^L_{MP}\Gamma^P_{NL} \tag{76}$$

$$[\nabla_M, \nabla_N]V_P = R_{MNP}{}^L V_L \tag{77}$$

Let us also write down the perturbed expressions

$$\Gamma^{(1)P}_{MN} = \frac{1}{2}(h^P{}_{M;N} + h^P{}_{N;M} - h_{MN;}{}^P) \tag{78}$$

$$R^{(1)}_{MN} = \frac{1}{2}(h^P{}_{M;NP} + h^P{}_{N;MP} - h^P{}_{P;MN} - h_{MN;P}{}^P) \tag{79}$$

For a Killing vector ξ^M, by definition we have $\xi_{M;N} + \xi_{N;M} = 0$, and from this we can derive the useful identity

$$\xi_{M;NP} = R_{MNP}{}^L \xi_L . \tag{80}$$

Another useful relation in deriving the surface expression (39) is

$$\xi^{R;M}\phi^{(0)}_{,R} + \xi^R \phi^{(0);M}{}_{,R} = \nabla^M(\xi^R \phi^{(0)}_{,R}) = 0 = \xi^R \phi^{(0)}_{,R} \tag{81}$$

The $D = 5$ Gamma matrices satisfy

$$\Gamma^{MNP}\Gamma_T = \Gamma^{MNP}{}_T + 3!\Gamma^{[MN}\delta^{P]}{}_T, \tag{82}$$

and repeated use of this identity leads to the useful formula

$$\Gamma^{MNP}\Gamma^{TS}R_{NPTS} = 4\Gamma^N G_N{}^M . \tag{83}$$

Let us also note the basic identity

$$[\nabla_M, \nabla_N] = \frac{1}{4}R_{MNAB}\Gamma^{AB} \tag{84}$$

and the useful expression

$$\Gamma^N\Gamma^M\Gamma^P\phi^{(0)}_{,N}\phi^{(0)}_{,P} = 2\phi^{(0),M}\Gamma^P\phi^{(0)}_{,P} - \Gamma^M\phi^{(0),P}\phi^{(0)}_{,P}. \tag{85}$$

For an action of the type

$$
\begin{aligned}
S_5 &= \int d^5x\sqrt{-g}\,[R - \frac{1}{2}(\partial\phi)^2 - \frac{c}{2\cdot 5!}e^{a\phi}F^2_{(5)}] \\
&\quad -T\int d^5x \int d^4\sigma\delta^5(x-X)\big[\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu X^M\partial_\nu X^N g_{MN}f(\phi(x)) - 2\sqrt{-\gamma} \\
&\qquad\qquad + \frac{2}{4!}\epsilon^{\mu\nu\rho\sigma}\partial_\mu X^M\partial_\nu X^N\partial_\rho X^P\partial_\sigma X^Q A_{MNPQ}(x)\big]
\end{aligned} \tag{86}
$$

where for simplicity we consider here just a single brane source, we have the equations of motion

$$
\begin{aligned}
G^{MN} &= \frac{1}{2}\phi^{,M}\phi^{,N} - \frac{1}{4}g^{MN}\phi^{,P}\phi_{,P} + \frac{c}{2\cdot 4!}e^{a\phi}(F^2)^{MN} - \frac{c}{4\cdot 5!}e^{a\phi}F^2 g^{MN} \\
&\quad -T\frac{1}{\sqrt{-g}}\int d^4\sigma\delta^5(x-X)\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu X^M\partial_\nu X^N f(\phi)
\end{aligned} \tag{87}
$$

$$
\begin{aligned}
\Box\phi &= \frac{ac}{2\cdot 5!}e^{a\phi}F^2 \\
&\quad +T\frac{1}{\sqrt{-g}}\int d^4\sigma\delta^5(x-X)\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu X^M\partial_\nu X^N g_{MN}\frac{\partial f}{\partial\phi}
\end{aligned} \tag{88}
$$

$$\nabla_M(e^{a\phi}F^{MNPQR}) = \frac{2T}{c\sqrt{-g}}\int d^4\sigma\delta^5(x-X)\epsilon^{\mu\nu\rho\tau}\partial_\mu X^N\partial_\nu X^P\partial_\rho X^Q\partial_\tau X^R \tag{89}$$

$$\gamma_{\mu\nu} = \partial_\mu X^M\partial_\nu X^N g_{MN}f(\phi) \tag{90}$$

$$
\begin{aligned}
0 &= \partial_\mu(\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\nu X^N f(\phi)) + \sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu X^P\partial_\nu X^Q\Gamma^N_{PQ}f(\phi) \\
&\quad -\frac{1}{2}\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu X^P\partial_\nu X^Q g_{PQ}\frac{\partial f}{\partial\phi}\phi^{,N} \\
&\quad -\frac{1}{4!}\epsilon^{\mu\nu\rho\tau}\partial_\mu X^P\partial_\nu X^Q\partial_\rho X^R\partial_\tau X^S F^N{}_{PQRS}
\end{aligned} \tag{91}
$$

Appendix B

For simplicity we will present the junction conditions here for the theory specified by the action (86), and we will work in the static gauge $X^\mu = \sigma^\mu$, $X^5 = 0, \pi$. Then there are no junction conditions associated with the 55 and $\mu 5$ components of the Einstein equations, since

$$T_{brane}^{55} = 0 = T_{brane}^{\mu 5}. \tag{92}$$

However $T_{brane}^{\mu\nu}$ is non-zero and singular, and we can derive the associated junction conditions by integrating the $\mu\nu$ components of the Einstein equations across the brane hypersurface. From

$$G_{\mu\nu} = -\frac{T}{\sqrt{g_{55}}}\delta(y)g_{\mu\nu}f^2(\phi) + Reg \tag{93}$$

we have

$$R_{\mu\nu} = \frac{T}{3\sqrt{g_{55}}}\delta(y)g_{\mu\nu}f^2(\phi) + Reg \; ; \tag{94}$$

then we integrate over the only component of the Ricci tensor that is singular, *i.e.* the component with two y derivatives. This is the only component that can give a singular contribution if the metric is continuous:

$$\int_{-\epsilon}^{+\epsilon} dy \; (-\frac{1}{2}g_{\mu\nu,yy}) = \int_{-\epsilon}^{+\epsilon} dy \; \frac{T}{3}\sqrt{g_{55}}\, g_{\mu\nu}f^2(\phi)\delta(y), \tag{95}$$

after which we take the limit $\epsilon \to 0$. Usually this would give us an expression for the jump in the normal derivative of the metric across the brane, but because of the $\mathbb{Z}_2$ symmetry that we are imposing about the brane, the normal derivative of the metric takes opposite values on opposite sides of the brane, and thus we find that not just its difference, but actually its total value is related to the value of the fields on the brane:

$$g_{\mu\nu,y}\big|_{y=0} = -\frac{T}{3}\sqrt{g_{55}}\, g_{\mu\nu}f^2(\phi)\big|_{y=0} \; . \tag{96}$$

A similar junction conditions can also be derived for the scalar field:

$$\phi_{,y}\big|_{y=0} = 2T\sqrt{g_{55}}\, f\frac{\partial f}{\partial \phi}\big|_{y=0} \; . \tag{97}$$

Appendix C

Here we reproduce the total action for the bulk plus two brane system written in the conventions of [14]; this system is equivalent to (18). Note that the brane action is written in

Nambu-Goto form:

$$
\begin{aligned}
S &= \int d^5x \sqrt{-g} \left(R - \frac{1}{2}(\partial\phi)^2 - V(\phi,x) \right) \\
&+ \frac{1}{6} \int d^5x \, \epsilon^{MNPQR} A_{MNPQ} \, \partial_R m(x) \\
&- 8m \int d^5x \sum_{i=1}^{2} s_i \int d^4\sigma \delta^5(x-X_i) \left(\sqrt{-\gamma} \, \tilde{W}(\phi(x)) + \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma} A_{\mu\nu\rho\sigma}(x) \right) , \quad (98)
\end{aligned}
$$

where $s_{1,2} = \pm 1$ as before and now

$$
V(\phi,x) = 8m(x)^2 \left(\tilde{W}(\phi)_{,\phi}^2 - \frac{2}{3}W'(\phi)^2 \right) \tag{99}
$$

$$
\tilde{W}(\phi(x)) = \frac{W}{2\sqrt{2}m} = e^{4\alpha\phi(x)} - \frac{5}{2}\sqrt{\frac{R_5}{20 \; m^2}} e^{\frac{8}{5}\alpha\phi(x)} \tag{100}
$$

Written in static gauge, the important equations of motion are:

$$
\partial_y m(y) = 2 \, m \, \delta(y) \tag{101}
$$

$$
F_{MNPQR} \equiv 5\partial_{[M}A_{NPQR]} = \sqrt{-g}\frac{V(x,\phi)}{2 \, m(x)}\epsilon_{MNPQR} . \tag{102}
$$

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